Graphs are abstract structures whose role is to model interactions between pairs of objects. Such objects are represented by the *vertices* of the graph, while the interactions are represented by the *edges* of the graph each connecting some two vertices. In general, the vertices of a graph can represent an arbitrary collection of objects, while the edges can represent an arbitrary relation that can be unambiguously said to be satisfied or not for any pair of objects from the collection. This generality, simplicity of the structures considered, and multitude of applications have made graph theory one of the most important and flourishing branches of finite mathematics. *Geometric intersection graphs* constitute a special class of graphs used to model geometric relations of certain types. In such graphs, the vertices represent objects of some geometric space, for example segments on a line (intervals) or in the plane, discs or rectangles in the plane, or balls in the three-dimensional space, while the edges connect pairs of vertices that intersect, that is, have some part in common.

Graph coloring is one of the oldest and most fundamental concepts considered in graph theory. To illustrate what it is about, we will make use of an example of a problem that was a driving force for the development of graph theory for decades, practically since its beginning—coloring of maps using four colors. Imagine a map on which individual countries are marked by border lines separating them. In order to make the map more legible, the countries should be colored so that any two countries sharing part of their borders are assigned different colors. It is not difficult to see that three color do not always suffice to color any map in this way. But are four colors always enough? This question has been raised in the 19th century, and only in the 1970s it has been proven (with the aid of computers) to have affirmative answer.

The map coloring problem is modeled in graph theory as follows. The vertices of the graph represent the countries, while the edges connect any two countries sharing part of their borders. Consequently, the task comes down to assigning colors to the vertices in such a way that any two vertices joined by an edge obtain distinct colors. A coloring with this property is called *proper*. It is worth noting that in general graph coloring problems, the *colors* can be values of any kind (e.g. numbers) and usually have nothing in common with real-life colors.

In many practical applications, graphs are used to model conflicts in access to some resource. In the simplest model, the problem of allocating the resources so that every demand is satisfied and simultaneously the minimum possible number of colors is used is exactly the problem of proper graph coloring. Here are some illustrating examples.

1. *Job scheduling*. A collection of jobs, each having their start and end times specified, are to be scheduled on a collection of identical machines so that each job gets assigned to one machine. Each machine can execute only one job at a time. Therefore, we look for a proper coloring of the graph in which every job is represented by a vertex, every conflict between two jobs (that is, a situation that two jobs cannot be assigned to the same machine) by an edge, and each machine by a color. It is not difficult to see that such a graph is simply an intersection graph of intervals—two jobs are in conflict when their intervals have some part in common.
2. *Frequency assignment in cellular networks*. We want to assign frequencies to cellular network transmitters in such a way that no region is covered by the signals from two transmitters using the same frequency. Therefore, we look for a proper coloring of the graph in which the vertices represent the transmitters, the edges represent the pairs of transmitters covering some common region, and the colors represent the frequencies. The region covered by the signal from each transmitter can be interpreted as a geometric object in the plane (e.g. a disc); the graph considered is then simply the intersection graph of these objects.

It is worth noting that typically, for applications of this kind, it is enough to find a coloring using "appropriately small" rather than optimal number of colors. This gives rise to the following questions: What graphs can be properly colored using few colors? The existence of what structures in the graph forces the use of many colors?

It is easy to see, that at least k colors are necessary to properly color a graph containing k mutually connected vertices, as each of them has to be assigned a distinct color. Such a set of k vertices of the graph is called a *clique*. A clique of size k is therefore the simplest possible structure forcing the use of k colors. On the other hand, even if the graph does not contain some specific structure such as a clique of size k, it can still require many colors. For example, there are graphs that require arbitrarily many colors but contain no *triangle*, that is, no clique of 3 vertices. A random (and hence almost every) n-vertex graph requires very many colors (of the order $n/\log n$) but does not contain even moderately small cliques (of the order $\log n$), provided that the number of vertices n is large enough.

However, if we restrict our considerations to geometric intersection graphs, the situation looks completely different. The condition that the graph has a geometric representation (which excludes random graphs) makes it possible to bound the required number of colors in terms of structural parameters of the graph such as the maximum clique size. For instance, exactly as many colors as is the maximum clique size are enough for intersection graphs of intervals. Such an equality is a quite unusual phenomenon for graph theory—it does not hold for the vast majority of geometric intersection graphs. Nevertheless, in many classes of such graphs, the required number of colors can still be bounded by some (even very large) function of the maximum clique size, and in other classes, it can be bounded by some small function of the number of vertices (of the order $\log n$ for graphs containing no triangles). These properties make geometric intersection graphs particularly interesting from the theoretic point of view in the context of coloring problems. This is the main reason for taking up this research topic in the present project. Those properties are important from the point of view of practical applications as well. For example, in the frequency assignment problem described before, it is reasonable to assume that the graph does not contain large cliques, for otherwise too many transmitters would be concentrated in one place; then, the above-mentioned theoretical results guarantee that a small number of frequencies is enough to avoid any conflicts.

Since the 1970s, when the problems of coloring graphs with geometric representations started to gain attention, it was believed that the required number of colors can be bounded in terms of the maximum clique size in arbitrary intersection graphs of connected objects in the plane. However, in 2012, a group of researchers (including the principal investigator of the present project) proved that this is not the case, devising a construction of intersection graphs of straight-line segments in the plane that require arbitrarily many colors while still containing no triangles. The essential ingredient in that work for the discovery of a relationship between geometric intersection graph coloring problems and on-line coloring problems for significantly simpler graphs. In on-line problems, the graph is constructed step by step by introducing new vertices and connecting them with some vertices introduced before; each such vertex has to be assigned its color immediately after it has been introduced. On-line coloring problems are important and interesting on their own, primarily because of their practical applications to modeling real-time systems (that is, systems making decisions without full knowledge of their consequences). Exploiting the relationship between ordinary colorings and on-line colorings has set the ground for many subsequent results that provide upper and lower estimates for the required number of colors in geometric intersection graphs. The present project also relies significantly on exploring and exploiting this relationship.

The principal objective of the project is to significantly improve the estimates on the required number of colors in broadest possible classes of geometric intersection graphs, in particular, for intersection graphs of straight-line segments in the plane. We want to achieve bounds of the order $\log \log n$ or $(\log \log n)^c$ (hence very small functions of the number of vertices n) instead of $\log n$. A bound of the order $\log \log n$ would be optimal, as this is the number of colors forced by the above-mentioned construction. On the other hand, we plan to search for new constructions, in particular, for ones forcing more than $\log \log n$ colors. We also plan to search for new class of geometric intersection graphs in which the required number of colors depends only on the maximum clique size, trying to outline the border between such classes and classes allowing the construction that forces $\log \log n$ colors. The project also aims at understanding the structure of geometric intersection graphs that require many colors but do not contain large cliques—for example, which other local substructures are enough to exclude in order to make the required number of colors bounded in terms of the maximum clique size.

The research carried out in the present project is mainly of theoretical importance. The minimum number of colors necessary for a proper coloring of a graph is one of most important and most extensively studied graph-theoretic parameters, but also, because of its non-local character, it is very difficult to compute or to estimate for the majority of graphs. Our research will help us understand its behavior in a class of graphs that is broad and important from both the theoretical and the practical points of view, which is in geometric intersection graphs. In particular, it will help us understand the influence of geometric representability on the possibility of bounding the required number of colors in terms of the structure of the graph. We also expect that the project results will help us find solutions to several other well-known graph-theoretic problems related to geometric intersection graphs.