Goal

The main goal of the project is a presentation of new complex approach to extremal axioms in mathematics. The analysis in question takes into account logical, mathematical and cognitive aspects of such axioms. The most important extremal axioms are: the completeness axiom in Hilbert's *Grundlagen der Geometrie*, the axiom of continuity, the axiom of induction, axioms of restriction in set theory and axioms of the existence of large cardinal numbers in set theory. All such axioms were thought of as conditions implying the (algebraic or semantic) uniqueness of the intended model of a theory. Usually, they express either maximality or minimality of a model. Their formulation was forced by mathematical discoveries in the XIXth century. The results in logic from the XXth century have shown some essential objective limitations of the deductive method. Recently one observes the process of sublimation of content of such axioms: one investigates their restricted versions (e.g. restricted induction), one points to different points of view in mathematics (e.g. Borel, constructive, and predicative point of view in set theory) and one analyzes their consequences for mathematical practice (e.g. the problem whether pathologies permissible by very general assumptions are responsible for incompleteness phenomena).

Research

The works concerning extremal axioms in general are not numerous, but there are many publications devoted to particular such axioms (continuity, induction, existence of large cardinal numbers). There is no monograph which describes these axioms from different points of view, related to their logical, mathematical and cognitive aspects. The final result of the present project will be such a monograph. It will discuss the difficulties connected with the characterization of the intended models of theories. Though there exists a large literature on the intended models of empirical theories, it does not reflect appropriately, in our opinion, the present state of knowledge about extremal axioms and their connections with the professional mathematicians' intuitions responsible for the context of discovery in mathematics. We will reveal these intuitions by an analysis of the source mathematical texts. The project proposes a new approach to mathematical intuitions, taking into account their stratification (elementary intuitions connected with human cognitive abilities, secondary intuitions imposed by the symbolic violence in the school, advanced intuitions of professional mathematicians) as well as results obtained in modern cognitive science and devoted to mathematical cognition.

Besides the analysis of the source mathematical texts which will result in a synthetic presentation of extremal axioms in a monograph form we will conduct some empirical research concerning the acquisition of mathematical intuitions. We will present a synthesis of observations collected during classes devoted to mathematical problem solving. We will organize a series of didactic experiments in which the cognitive behaviour of subjects involved in mathematical problem solving will be examined. Of special interest are such problem situations in which the subjects meet a necessity of applying new concepts and methods not known to them from the school.

The fundamental method applied in the theoretical part of the project is based on a critical analysis of the source mathematical texts, taking into account also results from the history of mathematics. We will consider first of all the views of professional mathematicians and the philosophers' proposals will be treated as auxilliary. Two facts deserve special attention. First, there exist subtle differences in the meaning of mathematical concepts used in different epochs, and therefore one should not impose the modern understanding of such concepts on their earlier understanding, which is to be recovered. Second, mathematical publications are the final products of the research and, as a rule, do not contain hints and commentaries what were the ways leading to the discoveries. The context of mathematical discovery is thus not overtly given in publications, one should reconstruct it.

The fundamental method in the empirical part of the project will be an active participation in the conducted didactic experiments. It consists of a preparation of the problem situation, observation of the subjects' cognitive behaviour, providing hints leading to successful strategies, correction of mistakes, etc. On the basis of such observations one will be able to formulate hints and directives concerning efficient teaching of mathematics at the university level.

Reasons

The main reason for an elaboration of the theoretical part of the project is a lack of a complex, holistic approach to the role of extremal axioms in the process of the creation of mathematical knowledge. We claim that extremal axioms played a crucial role in the development of mathematical thought. The situation in mathematics in the XIXth century was partly responsible for their formulation and the limitative results obtained in logic in the XXth century revealed some restrictions as far as a unique characterization of the intended models is concerned. This, in turn, implied a necessity of investigation of more sophisticated versions of such axioms together with their mathematical consequences. The presentation of these paths of development in the project will enable one to better understanding of the nature of mathematical cognition.

The main reason for conducting the empirical part of the project is to fill a serious gap in the mathematical education of our socjety. Almost all investigations devoted to the acquisition of mathematical intuitions are devoted to school education of children. Not enough attention is paid to the mathematical education of students (outside mathematics and information science). It often happens that their mathematical incompetence is caused by the traumatic experiences from the school. We claim that the results of the empirical part of the project may play a therapeutic role in this respect and that they will bring ideas how to shape the proper mathematical intuitions also in the case of adolescents.