Title of the project: Properties of solutions of nonlocal equations.

The first aim of this research project is to study fundamental properties of solutions of boundary value problems for some nonlocal operators in R^d. As an important example of a boundary value problem for nonlocal operators let us consider the following eigenvalue problem:

$$((-\Delta + m^2)^{(1/2)} - m)\, \varphi_n(x) = \lambda_n\, \varphi_n(x), \qquad \text{for x in D,}$$
$$\varphi_n(x) = 0, \qquad \text{for x in } D^c,$$

on a bounded convex domain D in R^d, where $m \geq 0$. Such eigenvalue problem appears in some models of relativistic quantum mechanics. It describes a quasi-relativistic particle in the box D. The nonlocal operator $H = (-\Delta + m^2)^{(1/2)} - m$ is often called Klein-Gordon square root operator or the (quasi)-relativistic Hamiltonian.

In this research project we plan to study fine properties of this eigenvalue problem such as p-concavity of the first eigenfunction φ_1 and the number of nodal domains of nth eigenfunction φ_n. Such problems are interesting both from mathatical and physical point of view. Obtaining p-concavity of φ_1 would give good bounds for the spectral gap $\lambda_2 - \lambda_1$. Such fine properties of this eigenvalue problem are well known in the classical case when $(-\Delta + m^2)^{(1/2)} - m$ is replaced by $-\Delta$.

We plan also to study this eigenvalue problem for more general nonlocal operators, where $(-\Delta + m^2)^{(1/2)} - m$ is replaced by -L, where L is a generator of a Lévy process in R^d. We also plan to investigate more genaral problems like e.g. $L \varphi = -1$ in a bounded convex domain D with Dirichlet condition $\varphi \equiv 0$ on D^c.

Another purpose of this project is to investigate the properties of solutions and spectral theory of the parabolic equations involving random nonlocal Schrödinger operators. We are interested in the initial value problems

$$\partial_t u(t,x) = L\, u(t,x) - V^\omega u(t,x), \qquad \text{for x in } R^d \text{ and } t > 0,$$
$$u(0,x) = u_0(x), \qquad \text{for x in } R^d,$$

where L is a generator of the symmetric Lévy process in R^d or the subordinate diffusion with fractal state space and V^ω is a stationary random field (called potential), mainly of the Poissonian type. The representative example to this class of operators is $L = -(-\Delta + m^2)^{(1/2)} + m$, $m \geq 0$, the generator of the so-called relativistic Lévy process. Such problems have been widely studied for classical random Schrödinger operators $-\Delta + V^\omega$ and lattice random operators.

We are going to study the annealed and quenched asymptotics of solutions u(t,x) when t tends to infinity, and the existence and asymptotics at the bottom of the spectrum (e.g. Lifschitz-type effect) of the integrated density of states (IDS) for nonlocal Schrödinger operators $-L+V^\omega$. We are also interested in the spatial asymptotics of u(t,x).

The parabolic random problem considered in this project can be translated to the evolution of a stochastic process in a random environment. Random motions in random media are an important subject in probability theory since there are a lot of applications to real-world problems in physics, chemistry and biology. For these reasons and also because of its mathematical interest, they have been studied a lot for decades, with a particular intensity in the last twenty five years. A physical ground for these mathematical investigations was made by P. W. Anderson, American physicist and Nobel laureate, who was the first one to suggest the possibility of electron localization inside a semiconductor, provided that the degree of randomness of the impurities or defects is sufficiently large. In his well known work, he observed a strong localization phenomena which can be described as the absence of diffusion of waves in a disordered medium.

Currently, one can observe an increasing interest in nonlocal models involving Schrödinger operators based on generators of jump Markov process. However, the delicate properties of nonlocal random operators and jump processes in random media are still very little understood. The problems we intend to investigate are motivated by those studied for classical operators. They are concerned with completely fundamental properties of operators appearing on the right hand side of the equation and corresponding jump processes. However, due to the nonlocal character of the equations they are much more difficult than their classical analogues.