The main goal of the Calculus of Variations is to find minima and maxima of such functions that depend on an infinite number of variables. The beginnings of the theory of Calculus of Variations date back to 1696, when Johann Bernoulli solved the problem of brachistochrony. The problem was to find a curve of the fastest descent, that would carry an idealized point-like body, starting at rest and moving along the curve, without friction, under constant gravity, to a given end point in the shortest time. This curve turned out ot be an arc of a cycloid. In the twentieth century Calculus of Variations experienced a rapid development, one could say that the main reason behind this was the heyday of functional analysis.

There is the following connection of Calculus of Variations with Partial Differential Equations. Suppose that we want to solve some partial differential equation $A[u]=0$. Moreover, lets assume that the operator A is the "derivative" of an appropriate functional I. Then our problem reads $I'[u]=0$. The advantage of this new formula is that we now can recognize solutions of the above equation as being critical points of I (i.a. minima and maxima). Of course, not every PDE can take this form, such class of equations is said to be *variational problems.*

Let us now consider the Dirichlet problem for the Laplace operator. The problem is to find a function $u:\Omega\rightarrow\boldsymbol{R}$ satisfying: $-\Delta u=0$ on Ω with $u=\varphi$ on the boundary of Ω.

In 1900, David Hilbert announced that he has solved the Dirichlet problem via the Dirichlet principle. Dirichlet's principle asserts that any solution u of that equation is a minimizer of the **Dirichlet integral**: $\boldsymbol{E}(v) = \int_\Omega |Dv|^2 dx$ in the class of functions $v:\Omega\rightarrow\boldsymbol{R}$ satisfying the boundary conditions $v=\varphi$ on the boundary of Ω.

It was pointed out that the proof was not rigorous. The touchy points where the following:

- Prove that $\boldsymbol{E}$ admits a minimizer;
- Prove that the minimizer is smooth and satisfies the equation $-\Delta u=0$.

It turned out that the announcement of Hilbert was a little premature. Instead, it became a program which stimulated many people for at least 40 years. The generalizations of this problem, due to its connections with the geometry and physics are investigated until today. Finally, in 1940 Hermann Weyl completed the proof, placing Calculus of Variations on a firm ground.

Starting from 1940's a number of mathematicians considered similar questions. Firstly, for systems of equations $u:\Omega\rightarrow\boldsymbol{R}^k$. Then, for mappings having values in an arbitrary k-dimensional manifold $\boldsymbol{N}$ (The reader, who is not familiar with the concept of manifolds, may try to imagine that $\boldsymbol{N}$ is a smooth surface without self-intersections, just like the surface of a sphere, or ellipsoid, or bicycle's inner tube).

To justify the notion of mappings having values in a manifold we will focus on the following example. Watching the map of an Airline company connections one can see that the trajectories of the airplanes look curved, which goes against our basic intuition, according to which the shortest path is a straight line. The reason behind this paradox is a simple geometrical fact: our earth is round and the shortest paths on a sphere are arcs of the great circles. Choosing the trajectories of airplanes is a simple illustration of a classical variational problem in geometry: finding the geodesic curves on a surface, namely paths on this surface with minimal lengths.

This considerations led to the notion of **harmonic mappings**. Let us consider once again the Dirichlet functional, this time for u with values in a manifold $\boldsymbol{N}$. Thanks to the work of John Nash, we know that such manifold $\boldsymbol{N}$ can be embedded into some Euclidean space $\boldsymbol{R}^N$ (for example, a two-dimensional sphere is a subset of the three-dimensional space $\boldsymbol{R}^3$). We need to specify the class of admissible maps. It seems reasonable to define the integral not only for smooth mappings, but also for mappings having integrable squares of their weak derivatives (The concept of weak derivatives is a generalization of the idea of differentiable functions. We extend this class to functions for which integration by parts is well defined), i.e. we assume that the map is in the Sobolev space $W^{1,2}$. Unfortunately, the Sobolev space with maps having values in a manifold does not posses very good properties, e.g. it is not linear. The sum of two mappings will generally not belong to the same space, simply because it does not have values in the same manifold. This seriously restricts the set of available methods that could be used in the study of properties of such mappings. It turns out that, due to the condition u having values in $\boldsymbol{N}$, the corresponding system of equations satisfied by harmonic mappings is nonlinear. In case $\boldsymbol{N}$ is a sphere it takes the form:

$$-\Delta u_i = |Du|^2 u_i \text{ dla } i=1,\ldots,m.$$

The properties of solutions to this system of equations vary very much than solutions to the Laplace equation. They are not necessarily minimizers of the energy, there is no uniqueness, in general, in the class of prescribed boundary conditions. Moreover, the solutions have singular points. There is a well-known example of an everywhere discontinuous harmonic map! However, the presence of the singularities is welcomed by the researchers, as the singularities of a nematic liquid crystal (harmonic maps with values in a sphere are a simplified model of nematic liquid crystals) are also observed in the nature.

The system of harmonic map equations have some special properties on flat domains. Namely, the Dirichlet energy is invariant under compositions with conformal mappings (A mapping is said to be conformal, if it locally preserves the angles). Harmonic mappings defined on the flat domains are continuous, the singular points appear only in higher dimensions.

Conformally invariant functionals are very important in modern mathematics. Finding conformally invariant functionals in dimension 4 led to the Hesse functional: $\boldsymbol{H}(u)=\int_\Omega|\Delta u|^2\, dx$. Here, the class of admissible maps are those with integrable squares of all second derivatives. Once again, we consider mappings having values in an arbitrary manifold $\boldsymbol{N}$. We say that a map is **biharmonic** if it is a critical point of the Hesse functional. This time, the corresponding system of equations is a system of equations of fourth order, whose solutions have many surprising properties: there is no uniqueness of solutions in the class of a prescribed boundary condition, the solutions do not have to be continuous and minimizing mappings are not the only critical points of the Hesse functional. Similarly, as in the corresponding harmonic case it is known that in the conformally invariant dimension, i.e. 4, all biharmonic maps must be continuous. In higher dimensions there may appear, not only point singularities but also, sets on which the map is not continuous.

The main objective of this project is to prove that in dimension higher than 4 there exists a neighborhood of the boundary, on which the minimizers with a prescribed boundary condition are continuous. One could imagine this in the following way. On the one hand, because of their structure on the macroscale, the boundary mapping can enforce the existence of singularities of the corresponding minimizer. On the other hand, due to the smoothness, the boundary mapping is repelling those singularities away from the boundary, just like an invisible pillow. Having this information would allow us to derive many important properties of minimizing biharmonic mappings. We expect to make a significant progress in construction of specific examples of biharmonic mappings, understanding the mechanisms governing the onset of singularities in dimension 5, as well as understanding how can the set of singular points look like.

One of the great desire of mathematician is to fully understand the behavior of solutions of partial differential equations, because of the presence of such equations in many problems of geometry and physics. Due to technical difficulties and other challenges this process takes a long time, whole decades. Similarly to the whole history of mathematics. The reason behind many theoretical studies is simply curiosity: the desire to understand a problem fully, or at least deeper than it was understood before, despite the efforts of other people. Each generation of mathematicians in this way make one's contribution to a huge construction of mathematics. This proposal is just one of the many projects of this kind: on the one hand simple, on the other hand - clearly being a part of the main issue in modern non-linear analysis and the theory of partial differential equations.