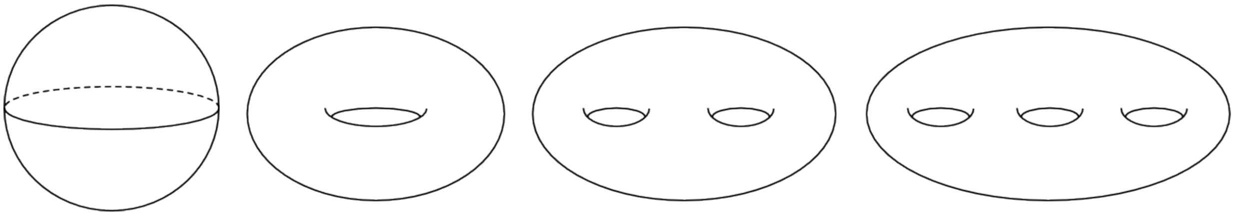
One of the most important common features of most applications of mathematics to other branches of science is the need to solve equations. On the whole, however, if we allow a very general form of equations, the problem is truly difficult, moreover the structure of solution set itself, for an equation in question, might be very complicated. For that reason mathematicians, in general, concentrate their efforts on very special classes of equations and the problem of finding solutions (which often is simply impossible to solve) is replaced by questions of a sort: *what are the properties of the solution set?, when do two equations have the same solution set?* etc.

The class of equations that is the base of our project are polynomial equations with two complex variables, i.e. equations like:

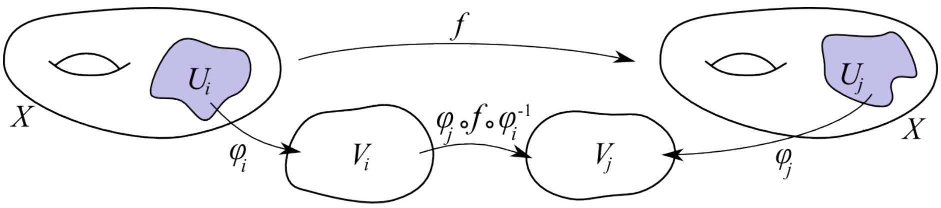
$$x^2+y^2+i=0, \quad y^2=x^2+x+1.$$

The common feature of all such equations is the fact that their solutions near non-singular points locally look like the complex plane. Hence if we restrict ourselves to non-singular equations and we perform the natural compactification (through the so called projectivization) of the solution sets, then the solutions we obtain shall be surfaces, which locally admit the same analytical structure as the complex plane. Such surfaces are called *Riemann surfaces* and up to continuous deformation they look like the surfaces below.



(From left to right we have respectively a sphere, torus, surface of genus 2, surface of genus 3 etc.) However, the picture above is misleading, as presenting surfaces up to a continuous deformation erases the information on the complex structure, which is integral part of a Riemann surface. What we see on the picture above is just a carrier of the structure (so called *topological type*) - on every surface of genus $g>0$ there exist infinitely many different complex structures and our project is mainly devoted to the studies of those structures.

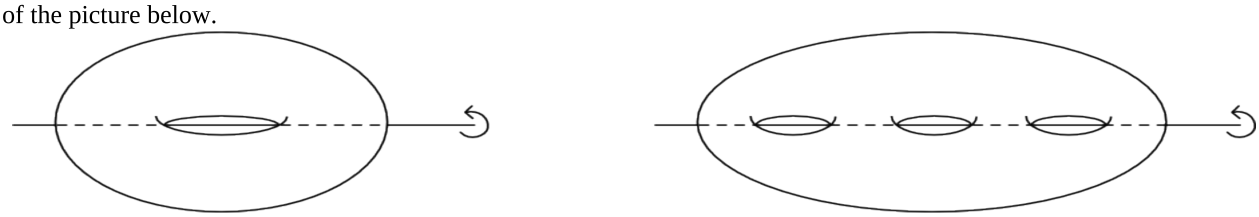
A very important notion, that distincts some of Riemann surfaces X is the one of *automorphism* $f:X\to X$, that is a bijective transformation of X which preserves the complex structure (meaning that after locally identifying the domain and image of the transformation with a subset of a complex plane, f is holomorphic).



Most of the Riemann surfaces admit no nontrivial automorphisms, so it is a very interesting problem, to describe or classify those Riemann surfaces, that actually have automorphisms. In the language of equations, Riemann surfaces with nontrivial automorphisms correspond to equations, which admit algebraic automorphism, that is one can employ a nontrivial algebraic change of variables in the equation in such a way, that the equation itself does not change (in the applications usually it means that the phenomenon described by the equation has sort of symmetry or conservation law). Such situation occurs for example with the equation:

$$y^2=x^2+x+1.$$

Here we can change the variables by substituting $(x',y')=(x,-y)$ and as a result the equation does not change. This means, that there is a nontrivial automorphism of the Riemann surface, which corresponds to the solution set of that equation. In this very example, the surface is a torus and the automorphism described above can be represented as a 180^o rotation around the axis seen in the left part of the picture below.



The notion that generalizes the example above are the so called *hyperelliptic* Riemann surfaces - these are the surfaces of genus possibly greater than 1, but admitting automorphism for which the geometry of the action is the same as on the picture above (i.e. the orbit space of the action is a sphere). Our project will be largely devoted to the studies of automorphisms of Riemann surfaces, for example we will deal with the problem of topological classification of possible actions on such surfaces.

Another interesting phenomenon is the fact, that in certain equations with complex coefficients one can change the variables in such a way, that the resulting equation has real coefficients (that is a so called real form of the equation we started with). For example in the equation: $x^2+y^2=i$, gdzie dokonując zamiany zmiennych:

$$(x',y')=(i\sqrt{i}x, i\sqrt{i}y) \text{ or } (x',y')=(\sqrt{i}x, \sqrt{i}y)$$

and the resulting equations are respectively:

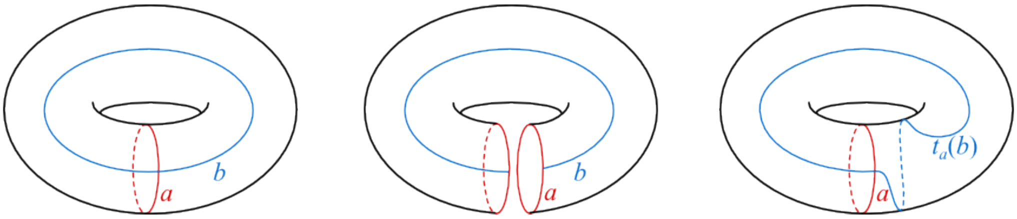
$$x^2+y^2=1 \text{ i } x^2+y^2=-1.$$

The existence of such a change of variables means, that the surface corresponding to the original equation's solution set has a symmetry (understood as an antiholomorphic automorphism). In such a way studying real forms of equations reduces to studying symmetries of the corresponding Riemann surfaces. In the example above the original equation's solution set is a sphere, and the substitutions we gave (and so the resulting equations with real coefficients) correspond to the two of its symmetries: one of which is the mirror reflection with respect to the plane containing the equator, and the other is the antipodal transformation. Let us also observe that the fixed point sets of these symmetries (that is, equator in the first case and empty set in the second case) correspond to the real solutions of real forms we obtained.

Another important problem which we shall study in the project is the problem of classification of Riemann surfaces of given genus g. It appears that all such surfaces can be parametrized by points of a very decent space M_g called the *moduli space*. In our project we shall study natural subsets S_g, R_g, H_g of this space, called respectively singular, real and hyperelliptic loci, which parametrize those Riemann surfaces which accordingly admit: nontrivial automorphism, symmetry or are hyperelliptic.

An important element in the construction of the moduli space M_g is the *mapping class group* (MCG in short). The group is formed of transformations of the fixed topological space, where we identify those transformations, which differ only by a continuous deformation. There are many examples known, when the algebraic properties of MCG allow to see interesting properties of the moduli space M_g. Let us mention the fact, that every transformation in the MCG can be obtained by composition of (executed in order) transformations of finite order means, that in the moduli space M_g there are no one-dimensional holes (it is simply connected). For that reason, studying of algebraic properties of MCG is very important and a great part of our project shall be devoted to that problem.

An example of a nontrivial element in MCG (that is surface transformation) is the so called *Dehn twist*, which is just a transformation t_a on the picture below - we cut the surface along the curve a, then we rotate one of the endings by 360º, and then we glue the cutting place back together.



Algebraic properties of the Dehn twists are important, because these elements geneate the MCG, that is any transformation f in MCG can be obtained by performing a finite number of twists. Such a presentation of f is not, however, unique and important question arises, asking about the possible relations between Dehn twists (as these relations are the source of decomposition ambiguity). An example of such a nontrivial relation is the *braid relation*: $t_at_bt_a=t_bt_at_b$, where t_a and t_b are Dehn twists along curves shown in the picture above. There are many different relations known between Dehn twists, nevertheless there are also still many open questions concerning them and we shall deal this problem in our project.

Another theme in our research shall be properties of three- and four-dimensional manifolds, that is spaces, which locally (in a small neighborhood of any point) look like standard three-dimensional space $\mathbf{R}^3$ or $\mathbf{R}^4$ instead of $\mathbf{R}^2$ as in the surface case. This subject, through its connections to theoretical physics for example, is studied intensively by mathematicians throughout the world for many years now. Our research of such manifolds concentrate on using properties of the surfaces (which are manifolds of dimension 2) to the construction of higher dimensional manifolds. There are many examples of such constructions, like Heegaard splittings, isogenous products, Lefschetz fibrations, open book decompositions. Not going into technical details, some classes of manifolds of dimension 3 and 4 can be studied by using 2-dimensional methods. In such a way our knowledge and experience acquired while studying mapping class groups and Riemann surfaces shall allow us to study objects potentially much more complicated.